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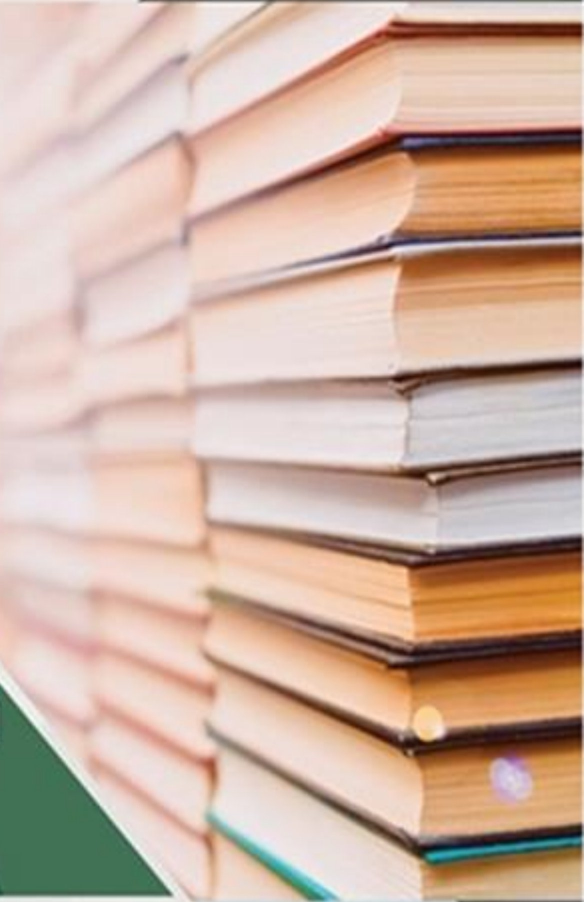

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ALMOST right (left) SEMICLEAN RINGS of SKEW GENERALIZED POWER SERIES

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Abstract

We extend the notions of almost clean, n-almost clean, and almost semiclean to the non-commutative setting. Then, we demonstrate that under specific conditions that the skew generalized power series rings $\mathcal{S}[[\mathcal{T},w]]$ is almost right (left) semiclean if and only if $\mathcal{S}$ is almost right (left) semiclean.

Keywords: Strictly ordered monoids; Artinan and narrow subsets; Generalized power series rings; Clean rings.

Mathematics Subject Classification 2010: 06F05, 16W60, 13A99.

1. Introduction

During the entire essay all rings $\mathcal{S}$ are associative not necessarily commutative with identity. An element that is not left (right) zero-divisor is called right (left) regular [15]. The set of left (right) regular elements is denoted by $Reg_l(\mathcal{S})$ $(Reg_r(\mathcal{S}))$. An element a in $\mathcal{S}$ is said to be n-potent if $a^n = a$. While an element $a \epsilon \mathcal{S}$ is called periodic if there exists a least positive integer m with $a^n = a^m$ such that $m < n$. The set of all periodic elements of $\mathcal{S}$ is denoted by $Per(\mathcal{S})$. If each element in a ring $\mathcal{S}$ is the sum of a unit and an idempotent. e (i.e. $e^2 = e \epsilon \mathcal{S}$), the ring is said to be clean.

The notion of clean rings was initiated by Nicholson in [13]. McGovern defined a commutative ring $\mathcal{S}$ to be almost clean in [11] if every element is the sum of a regular and an idempotent. The definition of n-almost clean rings given by the authors in [19] is as follows: a commutative ring $\mathcal{S}$ is said to be n-almost clean if every element $a \in \mathcal{S}$ is the sum of a regular and an n-potent.In [3], Anderson and Nitin characterized a commutative $\mathcal{S}$ to be almost semiclean if each element is the sum of a regular and a periodic. It is evident that the

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concept of an almost semi$c$lean ring generalizes the notion of an almost $c$lean. Purkait et al. studied noncommutative m-$c$lean rings in [14], and they further developed the concept of an m-exchange ring $\mathcal{S}$ where m-potents lift modulo every ideal of $\mathcal{S}$.

The motivation of this paper is twofold: Extending the concepts of almost $c$lean, n-almost $c$lean, and almost semi$c$lean properties defined for commutative rings to non-commutative rings is one of the two driving forces behind this work. The other is to show that under certain conditions the $s$kew $g$eneralized $p$ower $s$eries $r$ings $A = \mathcal{S}[[\mathcal{T}, w]]$ is almost right (left) semiclean if and only if $\mathcal{S}$ is almost right (left) semiclean. Also, we study the transfer of the m-$c$lean property from the ring $\mathcal{S}$ to the skew generalized power series $r$ings $A = \mathcal{S}[[\mathcal{T}, w]]$.

2. Almost right (left) Semi$c$lean Rings

In this section we extend the notions of almost $c$lean, n-almost $c$lean, and almost semi$c$lean rings to the non-commutative setting. Moreover, provide some examples.

Definition 2.1.

A ring $\mathcal{S}$ is called almost right (left) $c$lean if every element $a \in \mathcal{S}$ can be written as the sum of a right (left) regular element and an idempotent element.

Definition 2.2.

A ring $\mathcal{S}$ is called m- almost right (left) clean if every element $a \in \mathcal{S}$ can be written as the sum of right (left) regular and m- potent.

Definition 2.3.

A ring $\mathcal{S}$ is said to be almost right (left) semiclean if every element $a \in \mathcal{S}$ can be written as the sum of a right (left) regular element and a periodic element.

The following diagram explores the relations between the above-mentioned classes.

Clean	→	almost clean	→	almost right (left) clean
↓		↓		↓
n-clean	→	n-almost clean	→	m-almost right (left) clean
↓		↓		↓
Semiclean	→	almost semiclean	→	almost right (left) semiclean

In [3] Anderson and Bisht introduce examples to show that the above implications are irreversible.

Definition 2.4.

Let $1 < n < m$ be integers. A ring $\mathcal{S}$ is called (m,n)-indecomposable if for each $x \in \mathcal{S},\ x^m = x^n$ implies that $x = 0$ or $x = 1$. In particular a ring $\mathcal{S}$ is called indecomposable if $idem(\mathcal{S})$ = {0, 1}.

The following proposition showed that the classes almost right (left) $c$lean, m-almost right (left) $c$lean, and almost right (left) semi$c$lean are equivalent if $\mathcal{S}$ is an (m,n)-indecomposable ring.

Proposition 2.5.

Let $1 < n < m$ be integers. If $\mathcal{S}$ is an (m,n)-indecomposable ring. Then the following are equivalent:

(i) $\mathcal{S}$ is almost right (left) clean.

(ii) $\mathcal{S}$ is m-almost right (left) clean.

(iii) $\mathcal{S}$ is almost right (left) semiclean.

Proof:

(i) → (ii) → (iii) follows directly

(iii) → (i) Let $\mathcal{S}$ be an (m,n)-indecomposable almost right semi$c$lean. Then every element in $\mathcal{S}$ can be written as $x = r + f$ where $r \in Reg_r(\mathcal{S})$ and a periodic element f. Since $\mathcal{S}$ is an (m,n)-indecomposable, then $f = 0$ or 1, then $f \in Id\ (\mathcal{S})$. Hence, every element in $\mathcal{S}$ is the sum of right regular element and idempotent element. Therefore, $\mathcal{S}$ is almost right $c$lean.

The following proposition showed that classes clean, m-clean, semiclean, and almost right (left) semiclean are equivalent if $\mathcal{S}$ is a unit regular ring.

Proposition 2.6.

Let $\mathcal{S}$ be a unit regular ring then the following are equivalent:

(i) $\mathcal{S}$ is clean.

(ii) $\mathcal{S}$ is m-clean.

(iii) $\mathcal{S}$ is semiclean.

(iv) $\mathcal{S}$ is almost right (left) semiclean.

Proof:

(i) → (ii) → (iii) → (iv) Trivial.

(iv)→ (i). Let $\mathcal{S}$ be a unit regular almost right (left) semi$c$lean. Then $\mathcal{S}$ is clean [7, Theorem 5].

The following example shows that almost semiclean property is not right and left symmetric.

Example 2.7.

Let $\mathcal{S}$ be a ring of characteristic 2 consisting of the four elements $\{0, a, b, a + b\}$ where multiplication is defined by $a^2 = ab = a$ and $b^2 = ba = b$. $Per(\mathcal{S}) = \{f \in \mathcal{S} \mid f^3 = f^2\} = \{0, a, b\}$. Hence the set of $r$ight regular elements is empty. Consequently, we cannot write all elements of $\mathcal{S}$ as a sum of a $r$ight regular element and a periodic element. So, $\mathcal{S}$ is not almost $r$ight semi$c$lean ring, whereas $Reg_l(\mathcal{S}) = \{a, b\}$. Hence each $x \in \mathcal{S}$ can be written in the form $x = r + f$ where $r \in Reg_l(\mathcal{S})$ and $f^3 = f^2$. Therefore, $\mathcal{S}$ is almost $l$eft semi$c$lean ring which is not almost right semiclean.

3. Some properties of almost $r$ight ($l$eft) semi$c$lean rings

Remark 3.1.

While the homomorphic image of a semi$c$lean ring is semi$c$lean [20, Proposition 2.1], neither a homomorphic image nor a subring of an almost right (left) semi$c$lean ring is an almost right (left) semi$c$lean ring respectively [3, Example 3.1 and Example 3.3].

Proposition 3.2.

1. The direct product $\mathcal{S} = \Pi_{\alpha\in I}\mathcal{S}_\alpha$ of rings $\mathcal{S}_\alpha$ is an almost right (left) semi$c$lean ring if and only if each $\mathcal{S}_\alpha$ is almost right (left) semi$c$lean.

2. The direct product $\mathcal{S} = \Pi_{\alpha\in I}\mathcal{S}_\alpha$ of rings $\mathcal{S}_\alpha$ is an almost right (left) clean (m-almost right (left) $c$lean) if and only if each $\mathcal{S}_\alpha$ is almost right (left) $c$lean (m-almost right (left) clean).

Proof:

1. Suppose that $\mathcal{S} = \Pi_{\alpha\in I}\mathcal{S}_\alpha$ is an almost right semi$c$lean. Let the element (0, ..., a , 0, …,0) $\in \Pi_{\alpha\in I}\mathcal{S}_\alpha$ where $a \in \mathcal{S}_\beta$ and $\beta \in I$.Thus can be written as (0,..., a ,0, …,0)= $(r_\alpha)_\alpha + (x_\alpha)_\alpha$ with $(r_\alpha)_\alpha \in Reg_r(\mathcal{S})$ and $(x_\alpha)_\alpha^m = (x_\alpha)_\alpha^n$. Since $Reg_r(\mathcal{S}) = \prod Reg_r(\mathcal{S}_\alpha)$. $r_\alpha \in Reg_r(\mathcal{S}_\alpha)$ for each α and $x_\alpha^m = x_\alpha^n$.Therefore $a = r_\beta + x_\beta$ where $r_\beta \in Reg_r(\mathcal{S}_\beta)$ and $x_\beta^m = x_\beta^n$, consequently $\mathcal{S}_\beta$ is almost $r$ight semi$c$lean ring.

Conversely, assuming that each $\mathcal{S}_a$ is almost $r$ight semi$c$lean. Let $y = (y_\alpha)_{\alpha\in I} \in \Pi_{\alpha\in I}\mathcal{S}_\alpha$.Hence $y_\alpha = r_\alpha + x_\alpha$ for each $\alpha \in I$, where $r_\alpha \in Reg_r(\mathcal{S}_\alpha)$ and $x_\alpha^m = x_\alpha^n$.Then $y = (r_\alpha)_{\alpha\in I} + (x_\alpha)_{\alpha\in I}$ where $(r_\alpha)_{\alpha\in I} \in Reg_r(\Pi_{\alpha\in I}\mathcal{S}_\alpha)$ and $(x_\alpha)_{\alpha\in I}^m = (x_\alpha)_{\alpha\in I}^n$. So $(\Pi_{\alpha\in I}\mathcal{S}_\alpha)$ is an almost $r$ight semi$c$lean.

2. Similar to (1).

Lemma 3.3.

Every m-exchange ring is almost right (left) semi$c$lean.

Proof:

Since every m-exchange ring is m-potent [14, lemma 2.13], and every m-potent ring is m-$c$lean [14, lemma 2.2]. Hence,$\mathcal{S}$ is almost semi$c$lean.

It has been proved in [20, Theorem 4.1] that if e is an idempotent element in a ring $\mathcal{S}$ such that $e\mathcal{S}e$ and $(1 - e)\mathcal{S}(1 - e)$ are both semi$c$lean subrings then $\mathcal{S}$ is semi$c$lean.

The following example shows that a ring $\mathcal{S}$ is not almost semi$c$lean even if $\rho^{m-n}\ \mathcal{S}\ \rho^{m-n}$ and (1-ρ^{m-n}) $\mathcal{S}$ (1-ρ^{m-n}) are almost semi$c$lean where ρ be a periodic element in $\mathcal{S}$.

Example 3.4.

Let $\mathcal{S} = \mathcal{Z}_{15}$ = $\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7},\bar{8},\bar{9},\overline{10},\overline{11},\overline{12},\overline{13},\overline{14}\}$.Then, $Per\ (\mathcal{S})$ = $\{f^2 = f^3$: $f \in \mathcal{S}\}$ = $\{\bar{0},\bar{1},\bar{6},\overline{10},\}$and $Reg\ (\mathcal{S})$ = $\{\bar{1},\bar{2},\bar{4},\bar{7},\bar{8},\overline{11},\overline{13},\overline{14}\}$. Hence, $\bar{6}$ can not written as a sum of periodic element and a regular element. Consequently, $\mathcal{S}$ is not almost semi$c$lean ring. Since, ρ = $\bar{6}$ is a periodic element, it can be easily shown that $\rho\ \mathcal{S}\rho$ = $\{\ \rho\ r\ \rho\ :\ \in\ \mathcal{S}\ \}$ = $\{\ \bar{6}\ r\ \bar{6}: r\ \in\ \mathcal{S}\ \}$ = $\{\bar{0},\bar{3},\bar{6},\bar{9},\overline{12},\}$, where $Reg\ (\rho\ \mathcal{S}\ \rho\ )$ = $\{\bar{3},\bar{6},\bar{9},\overline{12},\}$ and $Per\ (\rho\ \mathcal{S}\ \rho\ )$ = $\{\bar{0},\bar{6}\}$ then every element in the subring $\rho\ \mathcal{S}\ \rho$ can be written as a sum of periodic element in $\rho\ \mathcal{S}\ \rho$ and a regular element in $\rho\ \mathcal{S}\ \rho$. Therefore, $\rho\ \mathcal{S}\ \rho$ is almost semi$c$lean.

Also, $(1-\ \rho)\ \mathcal{S}\ (1-\rho\ )$ = {-$\bar{5}$ r -$\bar{5}$: $r\ \in\ \mathcal{S}$} =$\{\bar{0},\bar{5},\overline{10}\}$, where $Reg(\ (\ 1-\rho)\ \mathcal{S}\ (\ 1-\rho\ ))$ = $\{\bar{5},\overline{10}\}$ and $per\ ((\ 1-\rho)\ \mathcal{S}\ (\ 1-\rho\ ))$ = $\{\bar{0},\overline{10}\}$. Then every element in the subring $(1-\ \rho)\ \mathcal{S}\ (1-\rho\ )$ can be written as a sum of periodic element in $(1-\ \rho)\ \mathcal{S}\ (1-\rho\ )$ and regular element in $(1-\ \rho)\ \mathcal{S}\ (1-\ \rho\ )$. Then $(1-\ \rho)\ \mathcal{S}\ (1-\rho\ )$ is almost semi$c$lean.

4. Skew Generalized Power Series Over Almost Right (Left) Semiclean Rings

Skew Generalized Power Series

Let $(\mathcal{T},.,\leq)$ be a strictly ordered monoid, $\mathcal{S}$ a ring, w:$\mathcal{T} \rightarrow End(\mathcal{S})$ a monoid homomorphism and w_s= $w(s)$ denotes the image of $s \in S$ under the action of w for any $s \in \mathcal{T}$. The set A of all maps $f: \mathcal{T}\ \rightarrow\ \mathcal{S}$ such that $supp\ (f)$=$\{s \in \mathcal{T}\ |f(s) \neq 0\}$ is artinian and $n$arrow subset of $\mathcal{T}$, with point-wise addition and a product operation known as convolution that is described by $(fg) = \sum_{(u,v)\in X_s(f,g)} f(u) w_u\left(g(v)\right)$ for each $f,g \in\ A$, where

$$X_s(f,g) = \{\ (u,v) \in \mathcal{T} \times \mathcal{T}\ |uv = s, f(u), and g(v) \neq 0\} \text{ is finite.}$$

With coefficients in $\mathcal{S}$ and exponents in S, $A = \mathcal{S}[[\mathcal{T},w]]$ thus becomes a ring known as skew $g$eneralized power series, with the identity map e: $\mathcal{T}\ \rightarrow R$ defined by e_1(1) =1 and e_1 (s)= · for each $1 \neq s\ \in\ \mathcal{T}$. The mapping $e_s \in A$ defined by e_s(s) = 1 and e_s(t) = 0 for each $t \neq s$ is an embedding of $\mathcal{T}$ into the monoid $(A,.)$. Also, $\mathcal{S}$ is canonically embedded as a subring of A via r⟶ c_r such that c_r(1) = r and c_r(s) = 0 for each $1 \neq s\ \in\ S$. So, we can identify $r\ \in \mathcal{S}$ with $c_r \in A$. Let $\pi\ (f)$ denote the set of all minimal elements of $supp(f)$. If $(\mathcal{T},\leq)$ is totally ordered, then $\pi\ (f)$ consists of only one element. To know more about the composition of $A = \mathcal{S}[[\mathcal{T},w]]$see [16],[17],[18],[1].

Remark 4.1 ([10]).

An ordered monoid $\mathcal{T}$ is said to be totally orderable if $(\mathcal{T},\leq)$ is an ordered monoid for some total order $\leq$. An ordered monoid $(\mathcal{T},\leq)$ is said to be quasitotally ordered, where $\leq$ is a quasitotal order on $\mathcal{T}$ if $\leq$ can be refined to an order $\preccurlyeq$ with respect to which $\mathcal{T}$ is a strictly,totally ordered monoid.

Proposition 4.2.

Let be a ring, $(\mathcal{T},\leq)$ a quasitotally ordered monoid, $w: \mathcal{T} \to End\,(\mathcal{S})$ a monoid homomorphism, and w_s compatible for every $s \in \mathcal{T}$. Assume that for every $f \in \mathcal{S}[[\mathcal{T},w]]$, there exists $s_o \in supp(f)$. If $f(s_o)$ is right (left) regular, then f is right (left) regular.

Proof:

Since $(\mathcal{T},.,\leq)$ is a quasitotally ordered monoid,then by hypothesis, the order $(\mathcal{T},\leq)$ can be refined to a strict total order $\preccurlyeq$ on $\mathcal{T}$. For every $f \in \mathcal{S}[[\mathcal{T},w]]$, there exists a unique minimal element of $supp(f)$ with respect to the total order $\preccurlyeq$, which will be denoted by $\pi_{\preccurlyeq}(f) = \{s_o\}$.

Suppose that $f(s_o)$ is $r$ight regular in R and there exist $0 \neq g \in \mathcal{S}\big[[\mathcal{T},w]\big]$ such that $f\,g = 0$ .Let $\pi_{\preccurlyeq}(g) = \{t_o\}$.For any $(u,v) \in X_{s_\circ t_\circ}(f,g), s_\circ \preccurlyeq u\,, t_\circ \preccurlyeq v$. If $s_\circ \prec u$, since $\preccurlyeq$ is a strict order, $s_\circ t_\circ \prec u t_\circ \preccurlyeq uv = s_\circ t_\circ$, a contradiction. Thus $u = s_\circ$. Similarly $v = t_\circ$.

Hence:$fg(s_\circ t_\circ) = \sum_{(u,v)\in X_{s_\circ t_\circ}(f,g)} f(u) w_u\big(f(v)\big) \; = f(s_\circ) w_{s_\circ}\big(g(t_\circ)\big) = 0$.

Using([12], lemma 3.1), it follows that $f(s_o)g(t_o) = 0$ which, contradicts the fact that $f(s_o)$ is a right regular element in R. Therefore $g(t_o) = 0$ which contradicts the fact that $\pi_{\preccurlyeq}(g) = \{t_o\}$ and $0 \neq g$ .Hence, f is a $r$ight regular element in $\mathcal{S}[[\mathcal{T},w]]$.

The converse of Proposition 4.2 is not true, for example.

Example 4.3.

Let $A = \mathrm{Z}_4[x]$ and $0 \neq f = 2 + x \in A$ .Then it can easily show that f is a regular element in A while $f(1)$ is a zero divisor.

Theorem 4.4.

Let $\mathcal{S}$ be a ring,$(\mathcal{T},\leq)$ a quasitotally ordered monoid, $w: \mathcal{T} \to End(\mathcal{S})$ a monoid homomorphism, and w_s compatible for every $s \in \mathcal{T}$. Then $A = \mathcal{S}[[\mathcal{T},w]]$ is almost right (left) semi$c$lean if and only if $\mathcal{S}$ is almost right (left) semi$c$lean.

Proof:

Since $(\mathcal{T},\leq)$ is a quasitotally ordered monoid, then by hypothesis, for every $f \in \mathcal{S}[[\mathcal{T},w]]$, there exists a unique minimal element of $supp(f)$ with respect to the total order $\preccurlyeq$, which will be denoted by $\pi_{\preccurlyeq}(f) = \{s_o\}$. Let $f \in A$ such that $\pi(f) = s_o$, $f(s_o) = a \in \mathcal{S}$. Since $\mathcal{S}$ is almost $r$ight semi$c$lean, then $a = r + b$ where, r is right regular and b is a periodic element.

Let $g = f - c_b e_{s_o}$. Then $g(s_o) = \big(f - c_b e_{s_o}\big)(s_o) = f(s_o) - c_b e_{s_o}(s_o)$

Since $X_{s_o}(c_b e_s) = \{(x,y) \in Supp(c_b) \bullet supp(e_s), : s_o = xy\} = \{(s_o,1)\}$

Thus $c_b e_s(s_o) = \sum_{(x,y)\in x_{s_o}} c_b(x) w_x e_s(y) = c_b(1) w_1\left(e_{s_o}(s_o)\right) = b\, w_1(1) = g(s_o)$ $= f(s_o) - c_b e_{s_o}(s_o) = f(s_o) - b = a - b = r + b - b = r$. Using Proposition 4.2, it follows that g is $r$ight regular in A. Since $b^m = b^n$, $\left(c_b e_{s_o}\right)^m ({s_o}^m) = \left(c_b e_{s_o}\right)\left(c_b e_{s_o}\right) \dots \left(c_b e_{s_o}\right)_{\text{m−times}} (s_o . s_o . . s_o)_{\text{m−times}} =$ $\left(c_b e_{s_o}\right)(s_o)\left(c_b e_{s_o}\right)(s_o) \dots \left(c_b e_{s_o}\right)(s_o) = c_b(1) w_1\left(e_{s_o}(s_o)\right) \dots c_b(1) w_1\left(e_{s_o}(s_o)\right) =$ $b.b \dots . b = b^m$, $\left(c_b e_{s_o}\right)^n ({s_o}^n) = \left(c_b e_{s_o}\right) \dots \left(c_b e_{s_o}\right)_{\text{n−times}} (s_o . s_o . \dots . s_o)_{\text{n−times}}$ $=\left(c_b e_{s_o}\right)(s_o)\left(c_b e_{s_o}\right)(s_o) \dots \left(c_b e_{s_o}\right)(s_o) = c_b(1) w_1\left(e_{s_o}(s_o)\right) \dots c_b(1) w_1\left(e_{s_o}(s_o)\right) = b \dots b =$ b^n. Then $\left(c_b e_{s_o}\right)^m = \left(c_b e_{s_o}\right)^n$. Thus $c_b e_{s_o}$ is a periodic element in A. Consequently, A is almost $r$ight semi$c$lean.

Conversely, let $a \in \mathcal{S}$ then $c_a \in A$ such that $c_a = f + g$ where f is right regular and g is a periodic element in A. $c_a(1) = (f + g)(1) = f(1) + g(1)$, then $a = f(1) + g(1)$ for each $a \in \mathcal{S}$. Since f is right regular. Then $f c_a \neq 0$. $f(1) w_1\left(c_a(1)\right) \neq 0$, then $f(1)a \neq 0$. Thus $f(1)$ is a $r$ight regular in $\mathcal{S}$. Since, g is periodic, $g^m = g^n$ such that $1 < m < n$. $g \cdot g \dots g$ (n −times) $= g \cdot g \dots g$ (m −times), $g \cdot g \dots g(1) = g \cdot g \dots g(1)$ $g(1) w_1\left(g(1)\right) w_1\left(g(1)\right) \dots \left(g(1)\right) = g(1) w_1\left(g(1)\right) \dots$ Then $(g(1))^m = (g(1))^n$. Thus $g(1)$ is a periodic element in $\mathcal{S}$. Hence, a is almost right semi$c$lean and $\mathcal{S}$ is almost right semi$c$lean.

Corollary 4.5.

Let $\mathcal{S}$ be a ring, $(\mathcal{T}, ., \leq)$ a quasitotally ordered monoid. Assume that for every $f \in \mathcal{S}[[\mathcal{T}, \leq]]$, there exists $s_o \in supp(f)$. If $f(s_o)$ is right (left) regular, then f is right (left) regular.

Proof:

Follows directly from Proposition 4.2 if w is the identity endomorphism for each $s \in S$.

Using the same technique used in the proof of Theorem 4.4 we can deduce the following result**:**

Theorem 4.6.

Let $\mathcal{S}$ be a ring, $(\mathcal{T}, \leq)$ is a quasitotally ordered monoid, $w: \mathcal{T} \to End(\mathcal{S})$ a monoid homomorphism, and w_s be compatible for every $s \in \mathcal{T}$. Then $A = \mathcal{S}[[\mathcal{T}, w]]$ is almost right (left) $c$lean (resp., m-almost right (left) $c$lean) if and only if $\mathcal{S}$ is almost right (left) $c$lean (resp., m-almost right (left) $c$lean).

If we suppose that w_s is the identity endomorphism for each $s \in \mathcal{T}$ in Proposition 4.2 and Theorem 4.6 respectively it can be easily shown the following Corollary.

Corollary 4.7.

Suppose that $\mathcal{S}$ is a ring,$(\mathcal{T},\leq)$ is a quasitotally ordered monoid. Then, the generalized power series ring $\mathcal{S}[[\mathcal{T},\leq]]$ is almost right (left) semiclean (resp., m-almost right (left) clean, almost right (left) clean) if and only if $\mathcal{S}$ is almost right (left) semi$c$lean (resp., m-almost right (left) clean, almost right (left) clean).

Proof:

Clear.

If we take $\mathcal{T}=\mathbb{N}\cup\{0\}$ with usual addition, usual order, and w_s is the identity for each $s\in\mathcal{T}$ we can deduce the following:

Corollary 4.8.

Suppose that $\mathcal{S}$ is a ring. Then, $\mathcal{S}[[x]]$ is almost right (left) semi$c$lean (resp., m-almost right (left) clean, almost right (left) $c$lean) if and only if R is almost right (left) semi$c$lean (resp., m-almost right (left) clean, almost right (left) clean).

Proof:

Follows directly from Theorem 4.4 and Theorem 4.6.

Now, if we take $\mathcal{T}=\mathbb{N}\cup\{0\}$ with usual addition, trivial $\leq$, and w_s is the identity endomorphism for each $\in\mathcal{T}$, The following conclusions can be drawn:

Corollary 4.9. ([3], Proposition 3.7)

The polynomial ring $\mathcal{S}[x]$ is almost $r$ight ($l$eft) $c$lean semiclean (resp., m-almost $r$ight ($l$eft) $c$lean, almost right (left) clean) ring if and only if $\mathcal{S}$ is almost $r$ight ($l$eft) $c$lean semi$c$lean (resp., m-almost right (left) clean, almost right (left) clean).

5. Skew Generalized Power Series over $\boldsymbol{m}$-clean rings

This section covers our study of some ring extensions o$v$er m-$c$lean rings.

Definition 5.1 ([14], 2.1)

Let $m\geq 2$ be a positive integer. An element x of a ring $\mathcal{S}$ is said to be m-$c$lean if x can be written as $x=u+f$, where u is a unit and f is an m-$p$otent element of R(i. e. $f^m=f$) . $\mathcal{S}$ is said to be m-$c$lean if every element of $\mathcal{S}$ is m-clean.

Proposition 5.2.

Let $m\geq 2$ be an integer. Then

1. An m-clean ring is its homomorphic image.

2. A $d$irect product of rings $\mathcal{S}=\prod\mathcal{S}_{\alpha\in I}$ is an m-$c$lean if and only if each $\mathcal{S}_\alpha$ is m-$c$lean.

Proof:

1. Since every element in the ring can be written as the sum of unit and m-potent. So, it is simple to demonstrate that a homomorphic image of an element, such as $x^m = x$, fulfills the same condition. A homomorphic image of a unit element is also unit.

2. (→) Follows from (1).

(←). Suppose that each $\mathcal{S}_\alpha$ is an m-clean. Let $y = (y_\alpha)_{\alpha\in I} \in \prod \mathcal{S}_{\alpha\in I}$ .Hence $y_\alpha = u_\alpha + x_\alpha$ for each $\alpha \in I$ where $u_\alpha \in U(\mathcal{S}_\alpha)$ and $x_\alpha^m = x_\alpha$ .Then $y = (u_\alpha)_{\alpha\in I} + (x_\alpha)_{\alpha\in I}$ where $(u_\alpha)_{\alpha\in I} \in U(\prod \mathcal{S}_{\alpha\in I})$ and $(x_\alpha)_{\alpha\in I}^m = (x_\alpha)_{\alpha\in I}$. So $(\prod \mathcal{S}_\alpha)$ is a m- $c$lean ring.

Theorem 5.3.

Let $\mathcal{S}$ be a ring, $(\mathcal{T}, \leq)$ a quasitotally ordered monoid, $w: \mathcal{T} \to End\,(\mathcal{S})$ a monoid homomorphism, and w_s be compatible for every $s \in \mathcal{T}$.Assume that for every $f \in A$ there exists $s_0 \in supp(f)$. If $s_0 \in U(\mathcal{T})$ and $\mathcal{S}$ is m-clean ring, then A is m-clean ring.

Proof:

Since $(\mathcal{T}, ., \leq)$ is a quasitotally ordered monoid, then by hypothesis, for every $f \in A$ there exists a unique minimal element of $supp(f)$ with respect to the total order $\preccurlyeq$, which will be denoted by $\pi_{\preccurlyeq}(f) = \{s_o\}$. Let $s_0 \in U(\mathcal{T})$ and $f(s_0) = a \in \mathcal{S}$, since $\mathcal{S}$ is m-clean, then $=$ $u + h$, where u is unit in $\mathcal{S}$ and h is m-potent (i.e., $h^m = h$). Let $g = f - c_h e_{s_0}$. Then $g(s_0) = \left(f - c_h e_{s_0}\right)(s_0) = f(s_0) - c_h e_{s_0}(s_0)$. Since, $c_h e_{s_o}(s_0) = \sum_{(x,y)\in X_{s_0}} c_h(x) w_x e_{s_o}(y)$ $=$ $c_h(1) w_1 \left(e_{s_o}(s_0)\right)$ $= h\, w_0(1) = h$, then $g(s_0) = f(s_0) - c_h e_{s_o}(s_0)$ $= a - h = u + h - h = u$. Since, $s_0 \in U(S)$ and $g(s_0) = u \in U(\mathcal{S})$. Then from ([8], 2.2), g is a unit in A.

$$\left(c_h e_{s_o}\right)^m (s_o{}^m) = \left(c_h e_{s_o}\right)\left(c_h e_{s_o}\right) \ldots \left(c_h e_{s_o}\right)_{\text{m−times}} (s_o . s_o .. s_o)_{\text{m−times}}$$

$$= \left(c_h e_{s_o}\right)(s_o)\left(c_h e_{s_o}\right)(s_o) \ldots \left(c_h e_{s_o}\right)(s_o) = c_h(1) w_1 \left(e_{s_o}(s_o)\right) \ldots c_h(1) w_1 \left(e_{s_o}(s_o)\right)$$

$= h.h \ldots .h = h^m$, $\left(c_h e_{s_0}\right)(s_0) = c_h(1)\, w_1 \left(e_{s_o}(s_0)\right) = h$. Then $\left(c_h e_{s_o}\right)^m = \left(c_h e_{s_o}\right)$. Thus $c_h e_{s_o}$ is m-potent in A. Consequently, A is m-clean.

Following the same procedure used in Corollary 4.8 we get:

Corollary 5.4.

Let $\mathcal{S}$ be a ring, $(\mathcal{T}, \leq)$ a quasitotally ordered monoid. Assume that for every $f \in \mathcal{S}[[\mathcal{T}, \leq]]$, there exists $s_0 \in supp(f)$ .If $s_0 \in U(\mathcal{T})$ and $\mathcal{S}$ is m-$c$lean, then $\mathcal{S}[[\mathcal{T}, \leq]]$ is m-clean.

Proof:

Follows easy using Theorem 5.3.

6. Conclusion:

In this paper we study the concepts of almost clean, n-almost clean, and almost semiclean in non-commutative rings. Then ,we proved that under specific conditions that the skew generalized power series rings $\mathcal{S}[[\mathcal{T},\leq]]$is almost right (left) semiclean if and only if $\mathcal{S}$ is almost right (left) semiclean. Also, we proved that the m-clean property transfer from the ring $\mathcal{S}$ to the skew generalized power series rings $\mathcal{S}[[\mathcal{T},\leq]]$.

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8. Conflict of Interests:

The authors clarify that they have no conflicts of interest.

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الملخص العربي

حلقات متسلسلة القوي المعممة المتخالفة الشبه نظيفة تقريبا يمني (يسري)

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الملخص العربي:

نقوم بعمل أمتداد لمفاهيم نظيفة تقريبا , n- نظيفة تقريبا وشبه نظيفة تقريبا إلى الحلقات الغير ابدالية. بعد ذلك، أثبتنا أنه تحت شروط محددة أن حلقات متسلسلة القوي المعممة المتخالفة $S[[\mathcal{T},w]]$ تكون شبه نظيفة تقريبا يمني (يسري) إذا و فقط اذا كانت S شبه نظيفة تقريبا يمني (يسري).