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Some extensions of right N-semilocal rings

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Abstract

In this paper, we study the necessary and sufficient conditions for the formal triangular matrix ring, the generalized upper triangular matrix ring, the trivial Morita context, and the Morita context to be right N-semilocal, right N-semiperfect, right N-right perfect, and right N-semiprimary.

Keywords: Formal triangular matrix ring; generalized upper triangular matrix ring; trivial Morita context; Morita context.

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1. Introduction

In our paper, $\mathcal{Q}$ is an associative ring with $1_{\mathcal{Q}}$, all modules are unitary, $\mathrm{J}(\mathcal{Q})$ is the Jacobson radical of $\mathcal{Q}$ and $\frac{\mathcal{Q}}{\mathrm{J}(\mathcal{Q})} = \bar{\mathcal{Q}}$.

A Morita context is $\mathcal{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$, where A, B are rings, ${}_{\mathrm{A}}\mathrm{P}_{\mathrm{B}}$ and ${}_{\mathrm{B}}\mathrm{Q}_{\mathrm{A}}$ are bimodules, and there exist context products $\mathrm{P} \times \mathrm{Q} \to \mathrm{A}$ and $\mathrm{Q} \times \mathrm{P} \to \mathrm{B}$ written multiplicatively as $(\omega, z) \mapsto \omega z$ and $(z, \omega) \mapsto z\omega$, such that $\mathcal{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$ is an associative ring with the obvious matrix operations. Morita first established Morita contexts in 1958 [7], and they were subsequently the topic of a lot of publications [1, 6, 7, 8, 9, 10, 11]. Some of examples of a Morita contexts are formal triangular matrix rings. In 1999, Haghany and Vardarjan [4] introduced the formal triangular matrix ring as follows: $\Delta = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ 0 & \mathrm{B} \end{bmatrix}$ has its elements formal matrices $\begin{bmatrix} \mathrm{a} & \mathrm{p} \\ 0 & \mathrm{b} \end{bmatrix}$ where $\mathrm{a} \in$

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$\mathbb{A}$, $b \in \mathbb{B}$ and $p \in P$ with usual definitions of matrix addition and multiplication. They proved that Δ is semilocal (resp. semiperfect, right perfect and semiprimary) if and only if $\mathbb{A}$ and $\mathbb{B}$ are semilocal (resp. semiperfect, right perfect and semiprimary). Varadarajan asked, what conditions must be met in order for a Morita context to be semilocal, semiperfect, and semiprimary. In 2014, Tang et al [11] answered these questions.

The authors in [2], take into account the family of the right N- semilocal rings that generalize the family of the semilocal rings, a ring Q is right N- semilocal if $\overline{Q}$ is right Noetherian, and right N-semilocal ring Q is called right N-semiperfect if idempotents lift modulo $J(Q)$. A right N-semilocal ring is right N-right (or left) perfect if $J(Q)$ is right (or left) T-nilpotent and is called right N-semiprimary if $J(Q)$ is nilpotent. The authors gave some examples to show that the family of right N-semilocal rings are a generalization of the family of semilocal rings and proved that the properties of right N- semilocal and right N-right perfect are Morita invariant. Fahmy et al in [3], proved that the finite normalizing extension S of Q is a right N-semilocal ring if and only if Q is a right N-semilocal ring. Also, they determined the requirements that must be met for the transfer of the right N-semilocal characteristics from a ring Q to the polynomial ring $Q[x]$ and vice versa.

The motivation of this paper is to extend the results of Varadarajan [4] and Tang [11] to be right N-semilocal, right N-semiperfect, right N-right perfect and right N-semiprimary. In part 2, we obtain necessary and sufficient conditions for a generalized upper (or lower) triangular matrix ring to be right N-semilocal, right N-semiperfect, right N-right perfect and right N-semiprimary. In part 3, the necessary and sufficient conditions for a Morita context that is a right N-semilocal, right N-semiperfect, right N-right perfect and right N-semiprimary are found in this part. Eventually, we investigate a Morita context with trivial context products.

A trivial Morita context is one such example of a Morita context. We define a right N-semilocal, right N-right perfect, and right N-semiprimary trivial Morita context.

Remark 1.1. If "left" is used instead of "right" throughout, then all the definitions, discussions, and the results remain the same as well. If the ring Q is an N-semilocal ring on both left and right, it is referred to as such.

2. The Generalized Triangular Matrix of Right N-semilocal Rings

This section includes the essential generalized matrix rings construction. One of the largest types of matrix rings is the generalized matrix ring, which has received significant studies in many fields.

First, we need the following definitions:

Definition 2.1. [2] A ring $\mathcal{Q}$ is right N-semilocal if $\bar{\mathcal{Q}}$ is right Noetherian.

Definition 2.2. [2] A ring $\mathcal{Q}$ is right N-semiperfect if $\bar{\mathcal{Q}}$ is right Noetherian and idempotents lift modulo $\mathrm{J}(\mathcal{Q})$.

Definition 2.3. [2] A ring $\mathcal{Q}$ is right N-right (or left) perfect if $\bar{\mathcal{Q}}$ is right Noetherian and $\mathrm{J}(\mathcal{Q})$ is right (or left) T-nilpotent.

Definition 2.4. [2] A ring $\mathcal{Q}$ is right N-semiprimary if $\bar{\mathcal{Q}}$ is right Noetherian and $\mathrm{J}(\mathcal{Q})$ is nilpotent.

In [5, page 65], let $\mathrm{A_i} = \mathrm{X_{ii}}$ be rings, and let $\mathrm{X_{ij}}$ be an $\mathrm{A_i} - \mathrm{A_j}$ $-$ bimodule for $\mathrm{i}, \mathrm{j} = 1,2, \ldots, \mathrm{n}$. Assume that bimodule homomorphisms exist

$$\mathrm{f_{ijk}} : \mathrm{X_{ij}} \otimes \mathrm{X_{jk}} \longrightarrow \mathrm{X_{ik}} \qquad \text{(I)}$$

Put

$$\mathrm{x_{ij}} . \mathrm{x_{jk}} = \mathrm{f_{ijk}}\, (\mathrm{x_{ij}} \otimes \mathrm{x_{jk}}) \qquad \text{(II)}$$

Additionally, assuming that bimodule morphisms (I) satisfies the associativity conditions:

$$(\mathrm{x_{ij}} . \mathrm{x_{jk}}) . \mathrm{x_{ks}} = \mathrm{x_{ij}} . (\mathrm{x_{jk}} . \mathrm{x_{ks}}) \qquad \text{(III)}$$

for all $\mathrm{x_{rt}} \in \mathrm{X_{rt}}$.

Consider the set

$$\mathcal{Q} = \begin{bmatrix} \mathrm{A_1} & \mathrm{X_{12}} & \cdots & \mathrm{X_{1n}} \\ \mathrm{X_{21}} & \mathrm{A_2} & \cdots & \mathrm{X_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{X_{n1}} & \mathrm{X_{n2}} & \cdots & \mathrm{A_n} \end{bmatrix} \qquad \text{(IV)}$$

In this set, addition is defined component wise, while multiplication is defined as in the typical matrix ring while accounting for (I) and (II). Following these operations, $\mathcal{Q}$ develops into a ring known as a generalized matrix ring of order n. A ring $\mathcal{Q}$ of the form (IV) is referred to as a formal triangular matrix ring or a generalized upper or lower triangular matrix ring. if $\mathrm{X_{ij}} = 0\ \forall\ 1 \leq \mathrm{j} < \mathrm{i} \leq \mathrm{n}$ (resp. $1 \leq \mathrm{i} < \mathrm{j} \leq \mathrm{n}$).

Proposition 2.5. [5, p. 69] Let

$$\mathcal{Q} = \begin{bmatrix} A_1 & X_{12} & \cdots & X_{1n} \\ 0 & A_2 & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \qquad \text{(V)}$$

be a generalized triangular matrix ring, where X_{ij} is an $A_i - A_j -$ bimodule, for $i, j = 1,2, \dots, n$. Then,

$$J(\mathcal{Q}) = \begin{bmatrix} J(A_1) & X_{12} & \cdots & X_{1n} \\ 0 & J(A_2) & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(A_n) \end{bmatrix} \qquad (V)^*$$

Proposition 2.6. [5, Corollary 2.6.16] Let $\mathcal{Q} = \begin{bmatrix} A_1 & X_{12} & \cdots & X_{1n} \\ 0 & A_2 & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$ be a generalized triangular matrix ring with the Jacobson radical $J(\mathcal{Q})$,where X_{ij} is an $A_i - A_j -$ bimodule, for $i, j = 1,2, \dots, n$. Then

1. $\mathcal{Q}/J(\mathcal{Q}) \cong A_1/J(A_1) \times \dots \times A_n/J(A_n)$.
2. Idempotents in $\mathcal{Q}$ can be lifted modulo the radical $J(\mathcal{Q})$ if and only if idempotents in A_i can be lifted modulo the Jacobson radical $J(A_i)$ for all $i = 1, \dots, n$.

Proposition 2.7. Let $\mathcal{Q} = \begin{bmatrix} A_1 & X_{12} & \cdots & X_{1n} \\ 0 & A_2 & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$ be a generalized triangular matrix ring with the Jacobson radical $J(\mathcal{Q})$, where X_{ij} is an $A_i - A_j -$ bimodule for $i, j = 1,2, \dots, n$. Then $\mathcal{Q}$ is a right perfect ring if and only if A_i are right perfect for all $i = 1, \dots, n$.

Proof. ($\Longrightarrow$) Suppose that $\mathcal{Q}$ is a right perfect ring with $J(\mathcal{Q})$ is the Jacobson radical, then $\mathcal{Q}/J(\mathcal{Q})$ is Artinian. Using Proposition 2.6, we have $A_i/J(A_i)$ is Artinian for all $i = 1,2, \dots, n$. Now, we need to show that $J(A_i)$ is right T -nilpotent ideals for all $i = 1,2, \dots, n$. Assume that we have m sequences $a_{i_1}, a_{i_2}, \dots, a_{i_m}, \dots \in J(A_i), i = 1, 2, \dots, n$. We have

$r_s = \begin{bmatrix} a_{1_s} & x_{12_s} & \cdots & x_{1n_s} \\ 0 & a_{2_s} & \cdots & x_{2n_s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_s} \end{bmatrix} \in J(\mathcal{Q})$ by proposition 2.5, $s = 1,2,\ldots.$ From the assumption, it follows that $J(\mathcal{Q})$ is right T-nilpotent, i.e., for any sequence $r_1, r_2, \ldots, r_n, \ldots$. There exists a positive integer k such that $r_k r_{k-1} \ldots r_1 = 0$.

Since $r_k r_{k-1} \ldots r_1 = \begin{bmatrix} a_{1_k} a_{1_{k-1}} \ldots a_{1_1} & x_{12}^{`} & \cdots & x_{1n}^{`} \\ 0 & a_{2_k} a_{2_{k-1}} \ldots a_{2_1} & \cdots & x_{2n}^{`} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_k} a_{n_{k-1}} \ldots a_{n_1} \end{bmatrix} =$

$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, we obtain that $a_{1_k} a_{1_{k-1}} \ldots a_{1_1} = 0$, … , $a_{n_k} a_{n_{k-1}} \ldots a_{n_1} = 0$, i.e., $J(A_i)$ are right T-nilpotent for each $i = 1,2, \ldots, n$. So, A_i are right perfect.

($\Longleftarrow$) Suppose that A_i be right perfect $i = 1, 2, \ldots, n$.. Then from Proposition 2.7, only need we show that $J(\mathcal{Q})$ is right T- nilpotent.

Let $r_s = \begin{bmatrix} a_{1_s} & x_{12_s} & \cdots & x_{1n_s} \\ 0 & a_{2_s} & \cdots & x_{2n_s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_s} \end{bmatrix} \in J(\mathcal{Q})$, $s = 1, 2, \ldots$. Using the assumption, it follows that $J(A_i)$ is right T-nilpotent, $i = 1, 2, \ldots, n$, i.e., for any sequence $a_{i_1}, a_{i_2}, \ldots, a_{i_l}, \ldots \in J(A_i)$, there exists $k_1 \geq 1$ such that $a_{i_{k_1}} a_{i_{k_1 - 1}} \ldots a_{i_1} = 0$, and there exists $k_s \geq k_{s-1} + 1$ such that $a_{i_{k_s}} a_{i_{k_s - 1}} \ldots a_{i_{k_{s-1}+1}} = 0, s = n, n-1, \ldots, 2.$ Thus we obtain $r_{k_n} \ldots r_1 = (r_{k_n} r_{k_n - 1} \ldots r_{k_{n-1}+1})(r_{k_{n-1}} r_{k_{n-1}-1} \ldots r_{k_{n-2}+1}) \ldots (r_{k_1} \ldots r_1)=$

$$\begin{bmatrix} a_{1_{k_n}} a_{1_{k_n - 1}} \ldots a_{1_{k_{n-1}+1}} & x_{12_n}^{`} & \cdots & x_{1n_n}^{`} \\ 0 & a_{2_{k_n}} a_{2_{k_n - 1}} \ldots a_{2_{k_{n-1}+1}} & \cdots & x_{2n_n}^{`} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_{k_n}} a_{n_{k_n - 1}} \ldots a_{n_{k_{n-1}+1}} \end{bmatrix} \ldots$$

$$\begin{bmatrix} a_{1_{k_1}} a_{1_{k_1 - 1}} \ldots a_{1_1} & x_{12_1}^{`} & \cdots & x_{1n_1}^{`} \\ 0 & a_{2_{k_1}} a_{2_{k_1 - 1}} \ldots a_{2_1} & \cdots & x_{2n_1}^{`} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n_{k_1}} a_{n_{k_1 - 1}} \ldots a_{n_1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & x\grave{}_{12_n} & \cdots & x\grave{}_{1n_n} \\ 0 & 0 & \cdots & x\grave{}_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & x\grave{}_{12_1} & \cdots & x\grave{}_{1n_1} \\ 0 & 0 & \cdots & x\grave{}_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = 0.$$

Therefore $\mathrm{J}(\mathcal{Q})$ is right T-nilpotent.

The necessary and sufficient needs for a generalized upper triangular matrix ring $\mathcal{Q}$ to be right N-semiprimary, right N-right (or left) perfect, right N-semiperfect, and right N-semilocal are presented in the following theorem.

Theorem 2.8. Let $\mathcal{Q} = \begin{bmatrix} A_1 & X_{12} & \cdots & X_{1n} \\ 0 & A_2 & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$ be a generalized triangular matrix ring with the Jacobson radical $\mathrm{J}(\mathcal{Q})$, where X_{ij} is an $A_i - A_j -$ bimodule for $i, j = 1,2, \dots, n$. Then

1. $\mathcal{Q}$ is a right N-semilocal if and only if A_i are right N-semilocal for all $i = 1, \dots, n$.
2. $\mathcal{Q}$ is a right N-semiperfect ring if and only if A_i are right N-semiperfect for all $i = 1, \dots, n$.
3. $\mathcal{Q}$ is a right N-semiprimary ring if and only if A_i are right N-semiprimary for all $i = 1, \dots, n$.
4. $\mathcal{Q}$ is a right N-right perfect ring if and only if A_i are right N-right perfect for all $i = 1, \dots, n$.

Proof. Statement 1 from Proposition 2.6 (1) it follows.

Statement 2 is a consequence of from Proposition 2.6 (2) and statement (1).

Statement 3: ($\Longrightarrow$) Suppose that $\mathcal{Q}$ is right N-semiprimary, then from definition $\mathcal{Q}/\mathrm{J}(\mathcal{Q})$ be right Noetherian and $\mathrm{J}(\mathcal{Q})$ is nilpotent. From proposition 2.6 (1), it follows that $A_i/\mathrm{J}(A_i)$ are right Noetherian for all $i = 1, \dots, n$. From proposition 2.5. It follows that $\mathrm{J}(A_i)$ are also nilpotent for all $i = 1, \dots, n$.

($\Longleftarrow$) Assume that A_i be right N-semiprimary for all $i = 1,2, \dots, n$. To show that $\mathcal{Q}$ is right N-semiprimary. Using statement 1, we proved that $\mathcal{Q}$ is right N-semilocal. Also, we will show that $\mathrm{J}(\mathcal{Q})$ is nilpotent. Since $\mathrm{J}(A_i)$ is nilpotent for all $i = 1,2, \dots, n$. We can find an integer $k \geq 1$ such that $\mathrm{J}(A_i)^k = 0$. $\mathrm{J}(\mathcal{Q})$ is immediately shown to be nilpotent.

Statement 4: this proof is clear from statement 1 and proposition 2.7.

We have the following result because a formal triangular matrix ring $\Delta = \begin{bmatrix} A & P \\ 0 & B \end{bmatrix}$, where P is $A - B$ bimodule, is a special case of the generalized triangular matrix ring.

Corollary 2.9. Let $\Delta = \begin{bmatrix} A & P \\ 0 & B \end{bmatrix}$ be a formal triangular matrix ring. Then

1. Δ is right N-semilocal if and only if A and B are right N-semilocal.
2. Δ is right N-semiperfect if and only if A and B are right N-semiperfect.
3. Δ is right N-semiprimay if and only if A and B are right N-semiprimary.
4. Δ is right N-right perfect if and only if A and B are right N-right perfect.

Example 2.10. [2, Example 2.2] Consider the bi-module $\mathbb{Q}$ of the rational numbers over the integer ring $\mathbb{Z}$ and take $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix} = \left\{ \begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{Z}, b \in \mathbb{Q} \right\}$. Then R is neither left or right Noetherian, since R has infinite ascending chains of left and right ideals of the form $\left\{ \begin{pmatrix} \mathbb{Z} & Q_p \\ 0 & 0 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 0 & Q_p \\ 0 & \mathbb{Z} \end{pmatrix} \right\}$, respectively, where $Q_p = \left\{ \frac{n}{p} : n \in Z, p \text{ is a fixed prime number} \right\}$. Since $\mathbb{Z}$ is Noetherian and $J(\mathbb{Z}) = 0$, it follows that $\mathbb{Z}$ is N-semilocal and N-semiprimary. Thus, from corollary 2.9, we get R is N-semilocal and N-semiprimary. Indeed, we have $J(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Q} \right\}$ and $\bar{R} \cong \mathbb{Z} \oplus \mathbb{Z}$ is Noetherian.

Recall that from [2] A right N-semilocal ring Q is right N-homogenous semilocal, weakly right N-homogenous semilocal, right N-local and weakly right N-local if $\bar{Q}$ is simple, prime, simple domain and domain, respectively.

Remark 2.11. For a generalized triangular matrix ring Q, since $Q/\mathrm{J}(Q) \cong \mathrm{A}_1/\mathrm{J}(\mathrm{A}_1) \times \ldots \times \mathrm{A}_\mathrm{n}/\mathrm{J}(\mathrm{A}_\mathrm{n})$ and the direct product of prime (resp. simple, domain) is not prime (resp. simple, domain), it follows that Q cannot be right N-homogenous semilocal, weakly right N-homogenous semilocal, right N-local and weakly right N-local.

3. General Case for Morita contexts

$Q = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$ is a general Morita context used throughout this section. We get necessary and sufficient conditions for Q to be right N-semilocal, right N-semiprimary, and right N-semiperfect in this section.”

Proposition 3.1. [11, Proposition 2.2.] Let $\mathcal{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$ be a Morita context. Then:

(1) $\mathcal{Q}_{\mathcal{Q}}$ is Artinian (resp. Noetherian) if and only if A_{A}, Q_{A}, P_{B} and B_{B} are Artinian (resp. Noetherian).

(2) ${}_{\mathcal{Q}}\mathcal{Q}$ is Artinian (resp. Noetherian) if and only if ${}_{\mathrm{A}}\mathrm{A}$, ${}_{\mathrm{B}}\mathrm{Q}$, ${}_{\mathrm{B}}\mathrm{B}$ and ${}_{\mathrm{A}}\mathrm{P}$ are Artinian (resp. Noetherian).

Theorem 3.2. [11, Theorem 2.5] Let $\mathcal{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$ be a Morita context with $\mathrm{J}(\mathcal{Q}) = \begin{bmatrix} \mathrm{J(A)} & \mathrm{P}_0 \\ \mathrm{Q}_0 & \mathrm{J(B)} \end{bmatrix}$, such that $\mathrm{P}_0 = \{\mathrm{x} \in \mathrm{P} : \mathrm{xQ} \subseteq \mathrm{J(A)}\}$ and $\mathrm{Q}_0 = \{\mathrm{y} \in \mathrm{Q} : \mathrm{yP} \subseteq \mathrm{J(B)}\}$.

Canonically, P/P_0 is an $(\mathrm{A}/\mathrm{J(A)}, \mathrm{B}/\mathrm{J(B)}) -$ bimodule and Q/Q_0 is an $(\mathrm{B}/\mathrm{J(B)}, \mathrm{A}/\mathrm{J(A)}) -$ bimodule, and a Morita context is created by it $\begin{bmatrix} \mathrm{A}/\mathrm{J(A)} & \mathrm{P}/\mathrm{P}_0 \\ \mathrm{Q}/\mathrm{Q}_0 & \mathrm{B}/\mathrm{J(B)} \end{bmatrix}$ the context products are provided by

$$(\mathrm{x} + \mathrm{P}_0)(\mathrm{y} + \mathrm{Q}_0) = \mathrm{xy} + \mathrm{J(A)}, (\mathrm{y} + \mathrm{Q}_0)(\mathrm{x} + \mathrm{P}_0) = \mathrm{yx} + \mathrm{J(B)}$$

For each $\mathrm{x} \in \mathrm{P}$ and $\mathrm{y} \in \mathrm{Q}$.

Proposition 3.3. [11, proposition 2.6] Let $\mathcal{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$ be a Morita context, and let $\begin{bmatrix} \mathrm{A}/\mathrm{J(A)} & \mathrm{P}/\mathrm{P}_0 \\ \mathrm{Q}/\mathrm{Q}_0 & \mathrm{B}/\mathrm{J(B)} \end{bmatrix}$ being described above. Subsequently, $\frac{\mathcal{Q}}{\mathrm{J}(\mathcal{Q})}$ is isomorphic to $\begin{bmatrix} \mathrm{A}/\mathrm{J(A)} & \mathrm{P}/\mathrm{P}_0 \\ \mathrm{Q}/\mathrm{Q}_0 & \mathrm{B}/\mathrm{J(B)} \end{bmatrix}$.

Theorem 3.4. Let $\mathcal{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$ be a Morita context. Then

(1) $\mathcal{Q}$ is right N-semilocal if and only if A, B are right N-semilocal and $(\mathrm{P}/\mathrm{P}_0)_{\mathrm{B}}$, $(\mathrm{Q}/\mathrm{Q}_0)_{\mathrm{A}}$ are right Noetherian.

(2) $\mathcal{Q}$ is right N-semiprimary if and only if A, B are right N-semiprimary and $(\mathrm{P}/\mathrm{P}_0)_{\mathrm{B}}$, $(\mathrm{Q}/\mathrm{Q}_0)_{\mathrm{A}}$ are right Noetherian.

Proof. (1) $\mathcal{Q}$ is right N-semilocal if and only if $\mathcal{Q}/\mathrm{J}(\mathcal{Q})$ is right Noetherian if and only if $\begin{bmatrix} A/J(A) & P/P_0 \\ Q/Q_0 & B/J(B) \end{bmatrix}$ is right Noetherian (by Proposition 3.3) if and only if $(A/J(A))_{A/J(A)}$, $(B/J(B))_{B/J(B)}$, $(P/P_0)_{B/J(B)}$, $(Q/Q_0)_{A/J(A)}$ are all right Noetherian (by Proposition 3.1) if and only if A and B are right N-semilocal and $(P/P_0)_B$, $(Q/Q_0)_A$ are right Noetherian.

(2) By using theorem 3.2, $J(\mathcal{Q}) = \begin{bmatrix} J(A) & P_0 \\ Q_0 & J(B) \end{bmatrix}$. If $J(A), J(B)$ are nilpotent, then

$\begin{bmatrix} J(A) & 0 \\ Q_0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & P_0 \\ 0 & J(B) \end{bmatrix}$ are left ideals of $J(\mathcal{Q})$ that are nilpotent, hence $J(\mathcal{Q})$ is nilpotent. On the other hand, since $J(\mathcal{Q})$ is nilpotent implies that $J(A)$ and $J(B)$ are also nilpotent. As a result, if $J(A)$ and $J(B)$ are nilpotent, $J(\mathcal{Q})$ must also be. This and (1) together prove the assertion.

Lemma 3.5. [11, lemma 3.2] Let $\mathcal{Q} = \begin{bmatrix} A & P \\ Q & B \end{bmatrix}$ be a Morita context with $PQ \subseteq J(A)$ and $QP \subseteq J(B)$. Each idempotent of $\mathcal{Q}/J(\mathcal{Q})$ is lifted to an idempotent of $\mathcal{Q}$, if A and B are rings for which idempotents lift modulo their Jacobson radicals.

Theorem 3.6. Let $\mathcal{Q} = \begin{bmatrix} A & P \\ Q & B \end{bmatrix}$ be a Morita context. If A and B are right N-semiperfect, then $\mathcal{Q}$ is right N-semiperfect.

Proof. It is directly from Theorem 3.4 (1) and Lemma 3.5.

4. Trivial Case for Morita Contexts

A Morita context $\mathcal{Q} = \begin{bmatrix} A & P \\ Q & B \end{bmatrix}$ is called trivial if the context products are trivial, i.e., $PQ = 0$ and $QP = 0$. In this section, we find necessary and sufficient conditions for a trivial Morita

Context to be right N-semilocal, right N-right perfect and right N-semiprimary.
The next result helps us to produce our result.

Lemma 4.1. [11, lemma 3.1] Let $\mathcal{Q} = \begin{bmatrix} A & P \\ Q & B \end{bmatrix}$ be a Morita context with $PQ \subseteq J(A)$ and $QP \subseteq J(B)$. Then the following statement hold:

The Jacobson radical of $\mathcal{Q}$ is $\begin{bmatrix} J(A) & P \\ Q & J(B) \end{bmatrix}$ and hence $\mathcal{Q}/J(\mathcal{Q})$ is isomorphic to $A/J(A) \times B/J(B)$.

Proposition 4.2. Let $Q = \begin{bmatrix} A & P \\ Q & B \end{bmatrix}$ be a trivial Morita context. Then

(1) Q is right N-semilocal if and only if A and B are right N-right semilocal.

(2) Q is right N-right Perfect if and only if A and B are right N-right Perfect.

(3) Q is right N-semiprimary if and only if A and B are right N-right semiprimary.

Proof. (1) Q is right N-semilocal ring with Jacobson radical $J(Q)$ iff $Q/J(Q)$ is right Noetherian iff $A/J(A) \times B/J(B)$ is right Noetherian (by Lemma 4.1) iff A, B are right N-semilocal.

(2) $\Longrightarrow$ From (1), we need only to show that $J(A)$ and $J(B)$ are right T-nilpotent ideals. It easy to see that $J(A)$ and $J(B)$ are right T-nilpotent. Therefor A and B are right N-right perfect.

$\Longleftarrow$ Conversely, suppose that both A and B are right N-right perfect. Then from (1) we just need showing that $J(Q)$ is right T- nilpotent.

Let $r_i = \begin{bmatrix} a_i & p_i \\ q_i & b_i \end{bmatrix} \in J(Q)$ for $i = 1, 2, \ldots$ since $J(A)$ and $J(B)$ are right T-nilpotent then for any sequence $a_i \in J(A)$ and for any sequence $b_i \in J(B)$ for $i = 1, 2, \ldots$ s. So, there exists $k_1 \geq 1$ such that $a_{k_1} \ldots a_1 = b_{k_1} \ldots b_1 = 0$, and there exists $k_2 \geq k_1 + 1$ such that $a_{k_2} \ldots a_{k_1+1} = b_{k_2} \ldots b_{k_1+1} = 0$. We obtain

$$r_{k_2} \ldots r_1 = (r_{k_2} \ldots r_{k_1+1})(r_{k_1} \ldots r_1) =$$
$$\begin{bmatrix} a_{k_2} a_{k_2-1} \ldots a_{k_1+1} & p_2 \\ q_2 & b_{k_2} b_{k_2-1} \ldots b_{k_1+1} \end{bmatrix} \begin{bmatrix} a_{k_1} a_{k_1-1} \ldots a_1 & p_1 \\ q_1 & b_{k_1} b_{k_1-1} \ldots b_1 \end{bmatrix} =$$
$$\begin{bmatrix} 0 & p_2 \\ q_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & p_1 \\ q_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ Therefore $J(Q)$ is right T-nilpotent.

(3) By Lemma 4.1, $J(Q) = \begin{bmatrix} J(A) & P \\ Q & J(B) \end{bmatrix}$. If $J(A)$ and $J(B)$ are nilpotent, then $\begin{bmatrix} J(A) & 0 \\ Q & 0 \end{bmatrix}, \begin{bmatrix} 0 & P \\ 0 & J(B) \end{bmatrix}$ are left ideals of $J(Q)$ that are nilpotent, hence $J(Q)$ is nilpotent. On the other hand, the fact that $J(Q)$ is nilpotent implies that $J(A)$ and $J(B)$ are also nilpotent. $J(A)$ and $J(B)$ have to be nilpotent in order for $J(Q)$ to be nilpotent. From this and (1), therefore the statement is true.

A formal triangular matrix ring is clearly a trivial Morita context, but Tang et al.'s example from [11] illustrates that a trivial Morita context does not have to be isomorphic to a formal triangular matrix ring. So, Proposition 4.2 generalizes the results of Corollary 2.9.

Example 4.3. [11, Example 3.9] Let $Q = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$ be trivial Morita context with the context's products coincide with those in $\mathbb{Z}_4$. Consequently, Q is not isomorphic to a formal triangular matrix ring and is indecomposable.

A Morita context $\begin{bmatrix} \mathrm{A} & \mathrm{P} \\ \mathrm{Q} & \mathrm{B} \end{bmatrix}$ with $\mathrm{PQ} \subseteq \mathrm{J(A)}$ and $\mathrm{QP} \subseteq \mathrm{J(B)}$ not necessarily isomorphic to any trivial Morita context.

Example 4.4. [11, Example 3.10] Let $Q = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$ be a Morita context, where the context products are identical to the product in $\mathbb{Z}_4$. Therefore, no trivial Morita context has an isomorphism with Q.

5. Conclusions

We showed that the family of right N-semilocal rings are a generalization of the family of semilocal rings and studied transfer the properties of right N- semilocal, right N-right perfect, right N- semiprimary, and right N- semiperfect between the formal triangular matrix ring, generalized matrix ring, Morita context, and trivial Morita context and their round rings.

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7. Conflict of Interests

The authors clarify that they have no conflicts of interest.

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الملخص العربي

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<u>الملخص العربي:</u>

في هذا البحث ، ندرس الشروط الضرورية والكافية لحلقة المصفوفات المثلثية ، وحلقة المصفوفات المثلثية العليا المعممة ، وسياق موريتا البديهي ، وسياق موريتا لتكون N- شبه محلية يمني و N- شبه مثالية يمني و N- صحيحة تامة يمني و N - شبه اساسية يمني.